

ASYMPTOTIC BEHAVIOUR OF THE SECTIONAL CURVATURE OF THE BERGMAN METRIC FOR ANNULI

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ABSTRACT. We extend and simplify results of [Din 2009] where the asymptotic behavior of the holomorphic sectional curvature of the Bergman metric in annuli is studied. Similarly as in [Din 2009] the description enables us to construct an infinitely connected planar domain (in our paper it is a Zalcman type domain) for which the supremum of the holomorphic sectional curvature is two whereas its infimum is equal to $-\infty$.

For a domain $D \subset \mathbb{C}^n$, $j = 0, 1, \dots$, $z \in D$, $X \in \mathbb{C}^n$ define

$$J_D^{(j)}(z; X) := \sup\{|f^{(j)}(z)(X)|^2 : f \in L_h^2(D), f(z) = 0, \dots, f^{(j-1)}(z) = 0, \|f\|_{L^2(D)} \leq 1\}.$$

Note that the functions above are the squares of operator norms of continuous operators defined on a closed subspace of $L_h^2(D)$.

Let us restrict ourselves to the case when D is bounded. Note that $J_D^{(0)}(z; X)$ is independent of $X \neq 0$ and is equal to the Bergman kernel $K_D(z, z)$. Moreover, we may express the Bergman metric as $\beta_D^2(z; X) = \frac{J_D^{(1)}(z; X)}{J_D^{(0)}(z; X)}$, $X \neq 0$. And finally the sectional curvature is given by the formula

$$R_D(z; X) = 2 - \frac{J_D^{(0)}(z; X)J_D^{(2)}(z; X)}{J_D^{(1)}(z; X)^2}, \quad X \neq 0.$$

Below we list a number of simple properties of the above functions.

The transformation formula for a biholomorphic mapping $F : D_1 \mapsto D_2$ is the following

$$J_{D_1}^{(j)}(z; X) = |\det F'(z)|^2 J_{D_2}^{(j)}(F(z); F'(z)X),$$

from which we get, among others, the independence of the sectional curvature for biholomorphic mappings $R_{D_1}(z; X) = R_{D_2}(F(z); F'(z)X)$.

If $D_1 \subset D_2$ then $J_{D_1}^{(j)} \geq J_{D_2}^{(j)}$.

We shall also need the continuity property of the functions just introduced with respect to the increasing family of domains.

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Proposition 1.

(1) Let D be a bounded domain in \mathbb{C}^n . Let $D = \bigcup_{\nu=1}^{\infty} D_{\nu}$ where $D_{\nu} \subset D_{\nu+1}$, D_{ν} is a domain in \mathbb{C}^n . Then for any j the sequence $(J_{D_{\nu}}^{(j)})_{\nu}$ is increasing and convergent locally uniformly on $D \times \mathbb{C}^n$ to $J_D^{(j)}$. In particular, the sequence $(\beta_{D_{\nu}})$ (respectively, $(R_{D_{\nu}})_{\nu}$) is locally uniformly convergent to β_D (respectively, R_D) on $D \times (\mathbb{C}^n \setminus \{0\})$.

(2) Let D be a bounded domain in \mathbb{C}^n . Assume that $D = \bigcup_{\nu=1}^{\infty} G_{\nu}$ where G_{ν} is a domain in \mathbb{C}^n . Assume additionally that for any compact set $K \subset D$ there is a ν_0 such that $K \subset G_{\nu}$ for any $\nu \geq \nu_0$. Then the sequence $(J_{G_{\nu}}^{(j)})_{\nu=1}^{\infty}$ is locally uniformly convergent to $J_D^{(j)}$. In particular, the sequence $(\beta_{G_{\nu}})$ (respectively, $(R_{G_{\nu}})$) is locally uniformly convergent to β_D (respectively, R_D) on $D \times (\mathbb{C}^n \setminus \{0\})$.

For a domain $D \subset \mathbb{C}$, $z \in D$ we put $J_D^{(j)}(z) := J_D^{(j)}(z; 1)$, $\beta_D(z) := \beta_D(z; 1)$, $R_D(z) := R_D(z; 1)$. Recall that $J_D^{(j)} = J_{D \setminus A}^{(j)}$ on $D \setminus A$ where A is a closed polar set in D such that $D \setminus A$ is connected.

Denote $P(\lambda_0, r, R) := \{\lambda \in \mathbb{C} : r < |\lambda - \lambda_0| < R\}$, $0 \leq r < R \leq \infty$, $\lambda_0 \in \mathbb{C}$. We also put $P(r, R) := P(0, r, R)$.

We are going to prove the following result.

Theorem 2. Let $r \in (0, 1)$, $\alpha \in (0, 1)$. Then

$$\begin{aligned} r^{2\alpha} J_{P(r,1)}^{(0)}(r^{\alpha}) &\sim \frac{1}{-\log r}, \quad r^{4\alpha} J_{P(r;1)}^{(1)}(r^{\alpha}) \sim \frac{2r^{2\alpha} + 2r^{2(1-\alpha)}}{1 - r^2}, \\ r^{6\alpha} J_{P(r;1)}^{(2)}(r^{\alpha}) &= \frac{A(r)}{B(r)}, \\ \text{where } A(r) &\sim \frac{r^2}{(1 - r^2)^2}(-2^4) + \frac{r^{6(1-\alpha)}}{(1 - r^2)(1 - r^4)}(A) + \frac{r^{6\alpha}}{(1 - r^2)(1 - r^4)}(-2^5), \\ B(r) &\sim \frac{2r^{2\alpha} + 2r^{2(1-\alpha)}}{1 - r^2} \end{aligned}$$

for some $A < -100$. The symbol $\varphi(r) \sim \psi(r)$ means that for any sufficiently small $\varepsilon > 0$ $\varphi(r) - \psi(r) = \psi(r)o(r^{\varepsilon})$.

In particular,

$$\begin{aligned} \lim_{r \rightarrow 0^+} R_{P(r;1)}(r^{\alpha}) &= -\infty \text{ for } \alpha \in (1/3, 2/3) \\ \lim_{r \rightarrow 0^+} R_{P(r;1)}(r^{\alpha}) &= 2 \text{ for } \alpha \in (0, 1/3] \cup [2/3, 1), \end{aligned}$$

The above theorem gives a generalization of a result from [Din 2009] (where the cases $\alpha = 1/2$, $\alpha = 0.3$ and $\alpha = 0.7$ have been done). It gives an answer to a problem posed in [Din 2009] on the asymptotic behavior of $R_{P(r;1)}(r^{\alpha})$ for arbitrary $\alpha \in (0, 1)$. Additionally, we present in Remark 4 the precise asymptotic behavior of $R_{P(r;1)}(r^{\alpha})$ as $r \rightarrow 0^+$.

Analogously as in [Din 2009] we may make use of Theorem 2 to construct an infinitely connected planar bounded domain with the supremum of the sectional curvature equal to 2 and its infimum equal to $-\infty$. The domain constructed by us is a Zalcman-type domain (unlike that in [Din 2009]) and the method of the proof of the above fact does not use, in contrast to [Din 2009], any sophisticated method.

Recall that the example from [Din 2009] (and certainly also the one presented in Corollary 3) may be seen as the final one in presenting examples where the supremum of the sectional curvature may be 2 (see [Chen-Lee 2009]) or its infimum may be equal to $-\infty$ (see [Her 2009]) – the example has simultaneously both properties.

Corollary 3. *Let $\theta \in (0, 1)$. Then there is a strictly increasing sequence $(n_k)_k$ of positive integers such that $\bar{\Delta}(\theta^{n_k}, \theta^{2n_k}) \cap \bar{\Delta}(\theta^{n_l}, \theta^{2n_l}) = \emptyset$, $k \neq l$, $\bar{\Delta}(\theta^{n_k}, \theta^{2n_k}) \subset \frac{1}{2}\mathbb{D}$ and*

$$\sup\{R_D(z) : z \in D\} = 2, \quad \inf\{R_D(z) : z \in D\} = -\infty,$$

where $D = \frac{1}{2}\mathbb{D} \setminus (\bigcup_{k=1}^{\infty} \bar{\Delta}(\theta^{n_k}, \theta^{2n_k}) \cup \{0\})$.

Proof of Theorem 2. We start with the analysis of some more general situation. For $0 < r < R$ denote $\alpha_n^{r,R} := \|\lambda^n\|_{P(r,R)}^2$, $n \in \mathbb{Z}$.

Note that

$$\frac{1}{2\pi} \alpha_n^{r,R} = \begin{cases} \frac{R^{2(n+1)} - r^{2(n+1)}}{2(n+1)}, & n \neq -1 \\ \log R - \log r, & n = -1 \end{cases}.$$

For $f \in L_h^2(P(r, R))$, $f(\lambda) = \sum_{n \in \mathbb{Z}} a_n \lambda^n$ the following identity

$$\|f\|_{P(r,R)}^2 = \sum_{n \in \mathbb{Z}} |a_n|^2 \alpha_n^{r,R}$$

holds.

Assume now that $r < 1 < R$.

Notice that

$$\begin{aligned} |f(1)|^2 &= \left| \sum_{n \in \mathbb{Z}} a_n \right|^2 = \left| \sum_{n \in \mathbb{Z}} a_n \sqrt{\alpha_n^{r,R}} \frac{1}{\sqrt{\alpha_n^{r,R}}} \right|^2 \leq \\ &= \sum_{n \in \mathbb{Z}} |a_n|^2 \alpha_n^{r,R} \sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n^{r,R}} = \|f\|_{P(r,R)}^2 \sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n^{r,R}}. \end{aligned}$$

Therefore, $J_{P(r,R)}^{(0)}(1) \leq \sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n^{r,R}}$.

In fact, the equality above holds – to see the equality it is sufficient to take $f \in L_h^2(P(r, R))$ with $a_n = \frac{1}{\alpha_n^{r,R}}$.

Our next aim is to give the formula for $J_{P(r,R)}^{(1)}(1)$ (which together with the previous one and general properties of the Bergman metric gives a formula for the Bergman metric of an arbitrary annulus at any point – see Remark 4).

We prove the equality

$$(1) \quad J_{P(r,R)}^{(1)}(1) = \sum_{n \in \mathbb{Z}} \frac{(n - \beta)^2}{\alpha_n^{r,R}},$$

for suitably chosen $\beta \in \mathbb{R}$ (to be given precisely later).

Let us start with $f \in L_h^2(P_{r,R})$ of the form $f(\lambda) = \sum_{n \in \mathbb{Z}} a_n \lambda^n$ such that $\sum_{n \in \mathbb{Z}} a_n = f(1) = 0$.

For such an f the following estimates hold

$$\begin{aligned} |f'(1)|^2 &= \left| \sum_{n \in \mathbb{Z}} n a_n \right|^2 = \left| \sum_{n \in \mathbb{Z}} (n - \beta) a_n \right|^2 = \left| \sum_{n \in \mathbb{Z}} \frac{n - \beta}{\sqrt{\alpha_n^{r,R}}} a_n \sqrt{\alpha_n^{r,R}} \right|^2 \leq \\ &= \sum_{n \in \mathbb{Z}} \frac{(n - \beta)^2}{\alpha_n^{r,R}} \sum_{n \in \mathbb{Z}} |a_n|^2 \alpha_n^{r,R} = \sum_{n \in \mathbb{Z}} \frac{(n - \beta)^2}{\alpha_n^{r,R}} \|f\|_{P(r,R)}^2. \end{aligned}$$

This gives the inequality ' \leq ' (with arbitrarily chosen β). Now we take f with $a_n = \frac{n - \beta}{\alpha_n^{r,R}}$, where β is such that the equality $\sum_{n \in \mathbb{Z}} a_n = f(1) = 0$ holds. If such a choice of β could be made we would get the equality in (1). But this means that we need to find β such that $\sum_{n \in \mathbb{Z}} \frac{n - \beta}{\alpha_n^{r,R}} = 0$, which however is satisfied exactly if

$$\beta = \frac{\sum_{n \in \mathbb{Z}} \frac{n}{\alpha_n^{r,R}}}{\sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n^{r,R}}}.$$

Consequently, with such a β we get the equality

$$\begin{aligned} J_{P(r,R)}^{(1)}(1) &= \sum_{n \in \mathbb{Z}} \frac{(n - \beta)^2}{\alpha_n^{r,R}} = \sum_{n \in \mathbb{Z}} \frac{n^2 - \beta n}{\alpha_n^{r,R}} + \beta \sum_{n \in \mathbb{Z}} \frac{\beta - n}{\alpha_n^{r,R}} = \\ &= \sum_{n \in \mathbb{Z}} \frac{n^2 - n\beta}{\alpha_n^{r,R}} = \frac{\varphi_{r,R}(2)\varphi_{r,R}(0) - \varphi_{r,R}(1)^2}{\varphi_{r,R}(0)}, \end{aligned}$$

where $\varphi_{r,R}(j) := \sum_{n \in \mathbb{Z}} \frac{n^j}{\alpha_n^{r,R}}$.

Let us now go on to the case of the annulus $P(r,1)$ where $0 < r < 1$. Our aim is to get the asymptotic behaviour of the curvature of $P(r,1)$ at r^α (for a fixed $\alpha \in (0,1)$) as $r \rightarrow 0^+$. First recall that

$$J_{P(r,1)}^{(j)}(r^\alpha) = r^{-2(j+1)\alpha} J_{P(r^{1-\alpha}, r^{-\alpha})}(1).$$

For simplicity we shall use the notation $\alpha_n = \alpha_n^{r^{1-\alpha}, r^{-\alpha}}$ and $J^{(j)}(1) = J_{P(r^{1-\alpha}, r^{-\alpha})}(1)$. Then we get the following formulas

$$\frac{\alpha_n}{2\pi} = \begin{cases} \frac{1 - r^{2(n+1)}}{2(n+1)r^{2(n+1)\alpha}}, & n \neq -1 \\ -\log r, & n = -1 \end{cases}.$$

From now on we forget about the constant 2π .

Note that for $n \geq 0$ the following formula holds

$$\alpha_{-n-2} = \frac{1 - r^{2(n+1)}}{2(n+1)r^{2(n+1)(1-\alpha)}}.$$

Let us define some functions (for $j = 0, 1, \dots$)

$$\begin{aligned} \varphi(j) &:= \sum_{n \in \mathbb{Z}} \frac{n^j}{\alpha_n} = \\ &= \frac{(-1)^j}{-\log r} + \sum_{n=0}^{\infty} \frac{2(n+1)}{1 - r^{2(n+1)}} (n^j r^{2(n+1)\alpha} + (-1)^j (n+2)^j r^{2(n+1)(1-\alpha)}) =: \frac{(-1)^j}{-\log r} + \psi(j). \end{aligned}$$

Then we may write the formula we have just obtained in the following form:

$$J^{(0)}(1) = \varphi(0), \quad J^{(1)}(1) = \frac{\varphi(2)\varphi(0) - \varphi(1)^2}{\varphi(0)}.$$

Note that the above formulas depend on r and α .

Our next aim is to find the formula for $J^{(2)}(1)$. We proceed similarly as above.

Let us start with $f \in \mathcal{O}(P(r^{1-\alpha}, r^{-\alpha}))$ with $f(\lambda) = \sum_{n \in \mathbb{Z}} a_n \lambda^n$ such that $\sum_{n \in \mathbb{Z}} a_n = f(1) = 0$ and $\sum_{n \in \mathbb{Z}} n a_n = f'(1) = 0$. Then

$$\begin{aligned} |f''(1)|^2 &= \left| \sum_{n \in \mathbb{Z}} n(n-1)a_n \right|^2 = \left| \sum_{n \in \mathbb{Z}} (n^2 - \beta n - \gamma)a_n \right|^2 = \\ &= \left| \sum_{n \in \mathbb{Z}} \frac{n^2 - \beta n - \gamma}{\sqrt{\alpha_n}} a_n \sqrt{\alpha_n} \right|^2 \leq \sum_{n \in \mathbb{Z}} \frac{(n^2 - \beta n - \gamma)^2}{\alpha_n} \sum_{n \in \mathbb{Z}} |a_n|^2 \alpha_n. \end{aligned}$$

As before if we find β, γ such that for $a_n = \frac{n^2 - \beta n - \gamma}{\alpha_n}$ the equalities $\sum_{n \in \mathbb{Z}} n a_n = \sum_{n \in \mathbb{Z}} a_n = 0$ hold then we shall have the equality

$$J^{(2)}(1) = \sum_{n \in \mathbb{Z}} \frac{(n^2 - \beta n - \gamma)^2}{\alpha_n} = \sum_{n \in \mathbb{Z}} \frac{n^2(n^2 - \beta n - \gamma)}{\alpha_n}.$$

The above properties are satisfied iff for some $\beta, \gamma \in \mathbb{R}$ the equalities

$$\begin{cases} \sum_{n \in \mathbb{Z}} \frac{n^2 - \beta n - \gamma}{\alpha_n} = 0 \\ \sum_{n \in \mathbb{Z}} n \frac{n^2 - \beta n - \gamma}{\alpha_n} = 0 \end{cases}$$

hold.

The above is equivalent to the following system

$$\begin{cases} \beta \sum_{n \in \mathbb{Z}} \frac{n}{\alpha_n} + \gamma \sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n} = \sum_{n \in \mathbb{Z}} \frac{n^2}{\alpha_n} \\ \beta \sum_{n \in \mathbb{Z}} \frac{n^2}{\alpha_n} + \gamma \sum_{n \in \mathbb{Z}} \frac{n}{\alpha_n} = \sum_{n \in \mathbb{Z}} \frac{n^3}{\alpha_n} \end{cases}$$

Since $\left(\sum_{n \in \mathbb{Z}} \frac{n}{\alpha_n}\right)^2 - \sum_{n \in \mathbb{Z}} \frac{n^2}{\alpha_n} \sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n} < 0$, the above system of equations has one solution

$$\begin{aligned} \beta &= \frac{\sum_{n \in \mathbb{Z}} \frac{n^2}{\alpha_n} \sum_{n \in \mathbb{Z}} \frac{n}{\alpha_n} - \sum_{n \in \mathbb{Z}} \frac{n^3}{\alpha_n} \sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n}}{\left(\sum_{n \in \mathbb{Z}} \frac{n}{\alpha_n}\right)^2 - \sum_{n \in \mathbb{Z}} \frac{n^2}{\alpha_n} \sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n}} = \frac{\varphi(2)\varphi(1) - \varphi(3)\varphi(0)}{\varphi(1)^2 - \varphi(2)\varphi(0)} \\ \gamma &= \frac{\sum_{n \in \mathbb{Z}} \frac{n}{\alpha_n} \sum_{n \in \mathbb{Z}} \frac{n^3}{\alpha_n} - \left(\sum_{n \in \mathbb{Z}} \frac{n^2}{\alpha_n}\right)^2}{\left(\sum_{n \in \mathbb{Z}} \frac{n}{\alpha_n}\right)^2 - \sum_{n \in \mathbb{Z}} \frac{n^2}{\alpha_n} \sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n}} = \frac{\varphi(1)\varphi(3) - \varphi(2)^2}{\varphi(1)^2 - \varphi(2)\varphi(0)} \end{aligned}$$

Therefore, we may write the formula

$$\begin{aligned} J^{(2)}(1) &= \varphi(4) - \beta\varphi(3) - \gamma\varphi(2) = \\ &= \frac{\varphi(4)\varphi(1)^2 - \varphi(4)\varphi(2)\varphi(0) - 2\varphi(3)\varphi(2)\varphi(1) + \varphi(3)^2\varphi(0) + \varphi(2)^3}{\varphi(1)^2 - \varphi(2)\varphi(0)}. \end{aligned}$$

So let us fix $\alpha \in (0, 1)$. Then for any $\varepsilon > 0$ small enough

$$\varphi(0) = \frac{1}{-\log r} + \frac{2r^{2\alpha}}{1-r^2} + \frac{2r^{2(1-\alpha)}}{1-r^2} + o(r^{2\alpha+\varepsilon}) + o(r^{2(1-\alpha)+\varepsilon}).$$

The asymptotic behaviour of $\varphi(1)^2 - \varphi(2)\varphi(0)$ is the following. The coefficients of the term of highest order (i.e. of $\frac{1}{(-\log r)^2}$) vanish and the term at $\frac{1}{-\log r}$ is the following

$$-(\psi(2) + \psi(0) + 2\psi(1)) = - \sum_{n=0}^{\infty} \frac{2(n+1)^3}{1-r^{2(n+1)}} (r^{2(n+1)\alpha} + r^{2(n+1)(1-\alpha)}).$$

The remaining terms are the following $\psi(1)^2 - \psi(2)\psi(0)$. Therefore, one may easily verify that the asymptotic behaviour is the following. For any $\varepsilon > 0$ small enough

$$\varphi(1)^2 - \varphi(2)\varphi(0) = \frac{1}{-\log r} \left(\frac{2r^{2\alpha}}{1-r^2} + \frac{2r^{2(1-\alpha)}}{1-r^2} \right) + o(r^{2\alpha+\varepsilon}) + o(r^{2(1-\alpha)+\varepsilon}).$$

We are remained with the asymptotic behavior of $\varphi(4)\varphi(1)^2 - \varphi(4)\varphi(2)\varphi(0) - 2\varphi(3)\varphi(2)\varphi(1) + \varphi(3)^2\varphi(0) + \varphi(2)^3$.

First note that the coefficients of the terms $\frac{1}{(-\log r)^j}$, $j = 2, 3$ vanish. On the other hand the coefficient of the term $\frac{1}{-\log r}$ is the following

$$\begin{aligned} & \psi(1)^2 - 2\psi(1)\psi(4) - \psi(4)\psi(2) - \psi(4)\psi(0) - \psi(2)\psi(0) \\ & - 2(-\psi(2)\psi(1) + \psi(3)\psi(1) - \psi(2)\psi(3)) + \psi^2(3) - 2\psi(0)\psi(3) + 3\psi(2)^2. \end{aligned}$$

Let us deal with the asymptotic behaviour (as $r \rightarrow 0$) of the last expression. One may calculate that for any $\varepsilon > 0$ small enough the last expression equals

$$\begin{aligned} & \frac{r^2}{(1-r^2)^2}(-2^4) + \frac{r^{6(1-\alpha)}}{(1-r^2)(1-r^4)}(A) + \\ & \frac{r^{6\alpha}}{(1-r^2)(1-r^4)}(-2^5) + o(r^2) + o(r^{6(1-\alpha)+\varepsilon}) + o(r^{6\alpha+\varepsilon}) \end{aligned}$$

where $A < -100$.

Combining all the obtained results we easily get the desired asymptotic behavior as claimed in the theorem. \square

Remark 4. Recall the formula for the curvature

$$R_{P(r,1)}(r^\alpha) = 2 - R(r, \alpha) := 2 - \frac{J^{(0)}(1)J^{(2)}(1)}{(J^{(1)}(1))^2}$$

Then the result of Theorem 2 gives, in particular, the asymptotic behavior of the expression $R(r, \alpha)$ (and consequently the asymptotic behaviour of the holomorphic curvature) as $r \rightarrow 0^+$ which looks as follows

$$\begin{cases} \frac{1}{-\log r} & \text{for } \alpha \in (0, 1/3] \\ \frac{1}{r^{6\alpha-2}(-\log r)} & \text{for } \alpha \in (1/3, 1/2] \\ \frac{1}{r^{6(1-\alpha)-2}(-\log r)} & \text{for } \alpha \in (1/2, 2/3) \\ \frac{1}{-\log r} & \text{for } \alpha \in [2/3, 1). \end{cases}$$

Remark 5. Note that in the proof of Theorem 2 we have obtained a formula for the Bergman kernel and metric in the annulus (compare [Her 1983], [Jar-Pfl 1993]) and a relatively simple expression for the sectional curvature of the annulus.

Proof of Corollary 3. We construct inductively sequences (n_k) , (x_k) , (y_k) and (r_k) such that $\theta^{n_1} + \theta^{2n_1} < x_1, y_1 < 1/2$ and for any $k = 1, 2, \dots$ the following properties hold: $\theta^{n_{k+1}} + \theta^{2n_{k+1}} < x_{k+1}, y_{k+1} < \theta^{n_k} - \theta^{2n_k}$, $\theta^{n_{k+1}} + \theta^{2n_{k+1}} < r_{k+1} < \theta^{n_k} - \theta^{2n_k}$ and for any compact $L \subset \bar{\Delta}(0, r_{k+1})$ for which $\Omega = \frac{1}{2}\mathbb{D} \setminus \left(\bigcup_{j=1}^k \bar{\Delta}(\theta^{n_j}, \theta^{2n_j}) \cup L\right)$ is connected the inequalities $R_\Omega(x_j) > 2 - 1/j$, $R_\Omega(y_j) < -j$ hold for any $j = 1, \dots, k+1$.

Then we put $D := \frac{1}{2}\mathbb{D} \setminus \left(\bigcup_{j=1}^\infty \bar{\Delta}(\theta^{n_j}, \theta^{2n_j}) \cup \{0\}\right)$. The properties we assumed ensure us that the domain D satisfies the inequalities $R_D(x_k) > 2 - 1/k$, $R_D(y_k) < -k$ which finishes the proof.

We go on to the construction of the above sequences. We put $r_1 := 1/4$. The possibility of the choice of n_1, x_1, y_1 as desired follows from Theorem 2 together with the biholomorphic invariance of the sectional curvature (we have to choose n_1 sufficiently large). The possibility of the choice of r_2 follows from Proposition 1. Now assume the system as above has been chosen for $j = 1, \dots, k$ (with the choice of $n_j, x_j, y_j, j = 1, \dots, k$ and $r_j, j = 1, \dots, k+1$).

First note that choosing $n_{k+1} > n_k$ so that $\theta^{n_{k+1}} + \theta^{2n_{k+1}} < r_{k+1}$ we get that the recursively defined set $D_{k+1} = \frac{1}{2}\mathbb{D} \setminus \left(\bigcup_{j=1}^{k+1} \bar{\Delta}(\theta^{n_j}, \theta^{2n_j})\right)$ satisfies the property $R_{D_{k+1}}(x_j) < -j$, $R_{D_{k+1}}(y_j) > 2 - \frac{1}{j}$, $j = 1, \dots, k$. Moreover, notice that after we choose n_{k+1} and x_{k+1}, y_{k+1} with $\theta^{n_{k+1}} + \theta^{2n_{k+1}} < x_{k+1}, y_{k+1} < \theta^{n_k} - \theta^{2n_k}$ and $R_{D_{k+1}}(x_{k+1}) > 2 - 1/(k+1)$, $R_{D_{k+1}}(y_{k+1}) < -(k+1)$ we easily get the existence of the desired r_{k+2} from Proposition 1. Therefore, what we need is to choose $n_{k+1} \gg n_k$ and properly chosen x_{k+1}, y_{k+1} . We choose x_{k+1}, y_{k+1} to be equal to $\theta^{n_{k+1}} + \theta^{\alpha 2n_{k+1}}$, where $\alpha = \frac{1}{2}$ in the case of x_{k+1} and $\alpha = \frac{1}{4}$ in the case of y_{k+1} .

We note that the following property holds:

For any small $\varepsilon > 0$, $\alpha \in (0, 1)$, $j = 0, 1, 2$ and for any $s \in (0, 1)$ there is an $0 < r_0 < s$ such that for any $0 < r < r_0$

$$(2) \quad 1 \leq \frac{J_{P(r,s)}^{(j)}(r^\alpha)}{J_{P(r,1)}^{(j)}(r^\alpha)} \leq r^{-\varepsilon}.$$

Actually, the left inequality is trivial. The right inequality can be proven as follows. First note that

$$J_{P(r,s)}^{(j)}(r^\alpha) = J_{P(\frac{r}{s}, 1)}^{(j)}\left(\frac{r^\alpha}{s}\right) s^{-2(j+1)}.$$

Since $\frac{r^\alpha}{s} = \left(\frac{r}{s}\right)^{\frac{\alpha \log r - \log s}{\log r - \log s}}$, the desired property follows from Theorem 2.

Note that

$$P(\theta^{n_{k+1}}, \theta^{2n_{k+1}}, r_{k+1} - \theta^{n_{k+1}}) \subset \frac{1}{2}\mathbb{D} \setminus \left(\bigcup_{j=1}^{k+1} \bar{\Delta}(\theta^{n_j}, \theta^{2n_j})\right) \subset P(\theta^{n_{k+1}}, \theta^{2n_{k+1}}, 1).$$

Making use of (2), Theorem 2 and the above inclusions we get the existence of n_{k+1} as claimed.

□

Remark 6. It would be interesting to find a precise description of Zalcman type domains having the property as stated in Corollary 3. Note that such a description (complete or at least partial) has been done for a description of the boundary behavior of the Bergman kernel, Bergman metric or Bergman completeness (see [Juc 2004], [Pfl-Zwo 2003], [Zwo 2002])

The construction presented in Corollary 3 is similar to the one presented in [Jar-Pfl-Zwo 2000] where the first example of a fat bounded planar domain which is not Bergman exhaustive has been presented.

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